Finite convergence of sum-of-squares hierarchies for the stability number of a graph

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Abstract. We investigate a hierarchy of semidefinite bounds $\vartheta^{(r)}(G)$ ($r \in \mathbb{N}$) for the stability number of a graph $\alpha(G)$, based on its copositive programming formulation. This hierarchy was introduced by de Klerk and Pasechnik [*SIAM J. Optim.* 12 (2002), pp.875–892], who conjectured convergence to $\alpha(G)$ in $r = \alpha(G) - 1$ steps. Even the weaker conjecture claiming convergence in finitely many steps is still open. We establish links between this hierarchy and sum-of-squares hierarchies based on the Motzkin-Straus formulation of $\alpha(G)$, which we use to show finite convergence when G is acritical (i.e., when $\alpha(G \setminus e) = \alpha(G)$ for all edges e of G). This result relies, in particular, on understanding the structure of the minimizers of the Motzkin-Straus formulation and showing that their number is finite precisely when G is acritical. As a byproduct we show that deciding whether a standard quadratic program has finitely many minimizers does not admit a polynomial-time algorithm unless P=NP. We also investigate the structure of the graphs satisfying $\vartheta^{(0)}(G) = \alpha(G)$. In particular, we give an algorithmic procedure that reduces the task of testing this property to the class of acritical graphs, and we show that a critical graph G has this property if and only if it can be covered by $\alpha(G)$ cliques.

Keywords: stability number of a graph, theta number, sum of squares of polynomials, semidefinite programming, copositive cone

Given a graph $G = (V, E)$, its *stability number* $\alpha(G)$ is defined as the largest cardinality of a stable set in G. Computing $\alpha(G)$ is a central problem in combinatorial optimization, well-known to be NP-hard (see for example [2]). A starting point to define hierarchies of approximations for the stability number is the following formulation by Motzkin and Straus [5], which expresses $\alpha(G)$ via quadratic optimization over the simplex $\Delta_n = \{x \in \mathbb{R}^n : x \geq 0, \sum_{i=1}^n x_i = 1\}$:

$$\frac{1}{\alpha(G)} = \min\{x^T(A_G + I)x : x \in \Delta_n\}, \tag{M-S}$$

where A_G is the adjacency matrix of G. Based on (M-S), de Klerk and Pasechnik [1] proposed the following copositive reformulation:

$$\alpha(G) = \min\{t : t(I + A_G) - J \in \mathrm{COP}_n\}.$$

Here, $\mathrm{COP}_n = \{M \in \mathcal{S}^n : x^T M x \geq 0 \, \forall x \in \mathbb{R}^n_+\}$ is the copositive cone, consisting of all copositive matrices. As linear optimization over COP_n is a hard problem, Parrilo [2000] introduced the following cones:

$$\mathcal{K}_n^{(r)} = \Big\{ M \in \mathcal{S}^n : \Big(\sum_{i=1}^n x_i^2 \Big)^r (x^{\circ 2})^T M x^{\circ 2} \in \Sigma \Big\},$$

where $x^{\circ 2} = (x_1^2, x_2^2, \ldots, x_n^2)$ and Σ is the set of sum-of-squares polynomials. The cones $\mathcal{K}_n^{(r)}$ provide sufficient conditions for matrix copositivity: for any integer $r \geq 0$ we have $\mathcal{K}_n^{(r)} \subseteq \mathcal{K}_n^{(r+1)} \subseteq \mathrm{COP}_n$. Moreover they cover the interior of the copositive cone:

$$\mathrm{int}(\mathrm{COP}_n) \subseteq \bigcup_{r \geq 0} \mathcal{K}_n^{(r)} \subseteq \mathrm{COP}_n.$$

De Klerk and Pasechnik [1] used the cones $\mathcal{K}_n^{(r)}$ to define the parameters:

$$\vartheta^{(r)}(G) = \min\{t : t(I + A_G) - J \in \mathcal{K}_n^{(r)}\},$$

which thus provide a hierarchy of upper bounds on the stability number, converging asymptotically to it: for any $r \geq 0$, we have $\alpha(G) \leq \vartheta^{(r+1)}(G) \leq \vartheta^{(r)}(G)$ and $\lim_{r \to \infty} \vartheta^{(r)}(G) = \alpha(G)$. The crucial property is that linear optimization over the cone $\mathcal{K}_n^{(r)}$ can be modelled as a semidefinite program and thus the parameter $\vartheta^{(r)}(G)$ can be computed using semidefinite optimization.

It is known that the parameter $\vartheta^{(0)}(G)$ coincides with $\vartheta'(G)$, the strengthening of the theta number $\vartheta(G)$ from Lovász [4], obtained by adding a nonnegativity constraint. Moreover, de Klerk and Pasechnik proved in [1] that $\vartheta^{(r)}(G) < \alpha(G) + 1$ for any $r \geq \alpha(G)^2$. So, it is possible to find $\alpha(G)$ by *rounding* after $\alpha(G)^2$ steps. On the other hand, they also conjectured finite convergence in $\alpha(G) - 1$ steps. This would imply that rounding is in fact not needed.

Conjecture 1 (De Klerk and Pasechnik [1]). Let G be a graph. Then we have $\vartheta^{(\alpha(G)-1)}(G) = \alpha(G)$.

However, it is not even known whether finite convergence holds at some step, i.e., whether $\vartheta^{(r)}(G) = \alpha(G)$ for some $r \in \mathbb{N}$.

Conjecture 2 (weaker). Let G be a graph. Then $\vartheta^{(r)} = \alpha(G)$ for some $r \in \mathbb{N}$.

Our first main result gives a partial positive answer to this second conjecture.

Theorem 1. *[3] Assume G is acritical, i.e., $\alpha(G \setminus e) = \alpha(G)$ for all edges e of G. Then finite convergence holds: $\vartheta^{(r)}(G) = \alpha(G)$ for some $r \in \mathbb{N}$.*

As a tool to show this result, we consider the Lasserre hierarchy for the problem (M-S): for an integer $r \geq 1$ define the parameter

$$f_G^{(r)} = \sup \Big\{ \lambda : x^T(A_G + I)x - \lambda = \sigma_0 + \textstyle\sum_{i=1}^n \sigma_i x_i + u(1 - \sum_{i=1}^n x_i) \\ \text{where } u \in \mathbb{R}[x], \ \sigma_0, \sigma_i \in \Sigma, \ \deg(\sigma_0), \ \deg(\sigma_i x_i) \leq 2r \Big\}.$$

As a first step we show the inequality

$$\frac{1}{\alpha(G)} \geq \frac{1}{\vartheta^{(2r)}(G)} \geq f_G^{(r+1)}$$

for all $r \geq 0$. The second key step is showing finite convergence of the parameters $f_G^{(r)}$ to $\frac{1}{\alpha(G)}$ for acritical graphs, which then directly implies Theorem 1. As a main ingredient, we characterize the set of minimizers of problem (M-S) and show that their number is finite if and only if the graph G is acritical.

The notion of criticality plays a crucial role in the study of these hierarchies. On the one hand, we can show finite convergence for the class of acritical graphs. On the other hand, it would suffice to show Conjectures 1 and 2 for the class of *critical* graphs, i.e., for the graphs satisfying $\alpha(G \setminus e) = \alpha(G) + 1$ for all edges e. In addition, we reduce the problem of deciding whether an edge e is acritical (i.e., $\alpha(G \setminus e) = \alpha(G)$) to the problem of deciding whether a quadratic program has finitely many minimizers. As a consequence, we obtain our second main result.

Theorem 2. *If there is a polynomial time algorithm to decide whether a standard quadratic program has finitely many global minimizers then $P = NP$.*

Graphs with $\vartheta^{(0)}(G) = \alpha(G)$

We also investigate the graphs for which the first relaxation $\vartheta^{(0)}(G)$ is exact. The study of these graphs is relevant to the question of understanding whether the basic semidefinite relaxation (also known as Shor relaxation) for polynomial optimization problems is exact. We characterize the critical graphs for which the first relaxation is exact.

Theorem 3. *Let G be a critical graph. Then, we have $\vartheta^{(0)}(G) = \alpha(G)$ if and only if G is the disjoint union of cliques.*

We also give a polynomial-time algorithmic procedure that reduces the question of deciding whether a graph satisfies $\vartheta^{(0)}(G) = \alpha(G)$ to the same question for acritical graphs.

Theorem 4. *For any fixed integer α, the problem of deciding whether a graph G with stability number α satisfy $\vartheta^{(0)}(G) = \alpha$ is reducible in polynomial time to the same problem for a graph with no critical edges.*

References

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