Polyhedral study of the Symmetrically Weighted Matrix Knapsack problem

Alexandre Heintzmann[1,2], Pascale Bendotti[1], and Cécile Rottner[1]

[1] EDF Lab Paris-Saclay, 7 Bd. Gaspard Monge, 91120 Palaiseau, France
{alexandre.heintzmann,pascale.bendotti,cecile.rottner}@edf.fr
[2] LAAS-CNRS, Université de Toulouse, CNRS, INP, Toulouse, France
{alexandre.heintzmann}@laas.fr

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1 Introduction

The knapsack problem and its variants have been widely studied in the literature [1]. Indeed, many problems can be seen as a knapsack type problem. We define and consider a variant of the knapsack problem defined as follows. Let M and N be positive integers. Consider a knapsack problem with N groups of M items. Let (i, j) be the item j of group i. Within each group, order constraints are such that any item (i, j) can be selected provided item $(i, j-1)$ is selected. Item (i, j) has value v_{ij} and weight w_j, which means that (i, j) and (i', j) have the same weight, thus the knapsack is symmetrically weighted with respect to the groups. The Symmetrically Weighted Matrix Knapsack problem (SWMK) is to maximizing the total value of the selected items, while the total weight is less or equal to C, the capacity of the knapsack. The motivation for this variant of the knapsack problem is that the SWMK is the core structure of the Hydro Unit Commitment, which is a production scheduling problem relative to hydroelectric units. Let x_{ij} be the binary variable such that $x_{ij} = 1$ if item (i, j) is selected in the solution. A formulation for the SWMK is the following

$$
\begin{aligned}
\max \sum_{i=1}^{N} \sum_{j=1}^{M} x_{ij} v_{ij} & \\
s.c. \sum_{i=1}^{N} \sum_{j=1}^{M} x_{ij} w_j \leq C & \\
x_{ij} \leq x_{ij-1} & \qquad \forall i, \forall j > 1 \\
x_{ij} \in \{0, 1\} & \qquad \forall i, \forall j
\end{aligned}
$$

The SWMK is by definition a special case of the precedence-constrained knapsack problem [3]. Even if there are several groups of items, the SWMK is not a multiple knapsack problem [4]. Indeed, the SWMK has a single global capacity constraint, applied on every group. The presented SWMK is with order constraints. Instead a formulation for the SWMK can be derived with disjunctive constraints. Order constraints and disjunctive constraints lead to formulations with the same LP relaxation [6]. In the general case, the SWMK is a generalization of the integer knapsack problem [5]. As the integer knapsack problem is NP-hard, then the SWMK is also NP-hard.

2 Polyhedral study

From the convex hull, we distinguish three types of inequalities: inequalities from the formulation (bounds, order, ...), inequalities with coefficients 0-1 called binary inequalities, and inequalities with positive integer coefficients called integer inequalities. Some bounds and the order inequalities are proved to be facet-defining. In the article, the work focuses on finding necessary and sufficient conditions for the binary inequalities to be facet-defining. The families of inequalities from the knapsack [2] and the precedence-constrained knapsack [3] do not contain the binary inequalities of the SWMK.

In term of dimension, when any item can be selected in a feasible solution, the polyhedron is proven to be full dimensional. Also, if for an instance A of the SWMK, an item cannot be selected in a feasible solution, it is possible to create an instance B with fewer items, that has the exact same feasible solutions as A, and for which any item can be selected in a feasible solution.

When an inequality is facet-defining for the SWMK, any permutation of the group indices leads to another facet-defining inequality. Thus, for every facet-defining inequality of the SWMK, there is an exponential number of symmetric facet-defining inequalities. In order to reduce the number of inequalities to be handled, we define patterns.

Definition 1 (Pattern $\mathcal{P}$). *A pattern is a set of N sets $S_i \subseteq \{1, ..., M\}$, $i \leq N$. Each set S_i contains the indices j of the items in group i.*

As the sets are not ordered, one pattern can represent every permutation of group indices.

Definition 2 (Variable set $\mathcal{X}$ associated to $\mathcal{P}$). *A variable set $\mathcal{X}$ is associated to pattern $\mathcal{P}$ if for a given permutation π of $\{1, .., N\}$: $x_{ij} \in \mathcal{X} \Leftrightarrow j \in S_{\pi(i)} \in \mathcal{P}$.*

Definition 3 ($rank(\mathcal{P})$). *The $rank(\mathcal{P})$ is the valid upper bound for the sum of variables in any set $\mathcal{X}$ associated to $\mathcal{P}$.*

Definition 4 (Pattern inequalities). *The pattern inequalities of a pattern $\mathcal{P}$ are as follows, with $\mathcal{X}$ one of the sets of variables associated to $\mathcal{P}$:*

$$\sum_{x_{ij} \in \mathcal{X}} x_{ij} \leq rank(\mathcal{P})$$

Theorem 1. *Let $\mathcal{P}$ be a pattern where each set S_i has cardinality at most one and with $rank(\mathcal{P}) = p$. The following conditions are necessary and sufficient for pattern inequalities of pattern $\mathcal{P}$ to be facet-defining with $\mathcal{X}$ a set of variables associated to $\mathcal{P}$.*

$$\text{(i)} \qquad |S_i| \geq 1 \qquad \forall S_i \in \mathcal{P}$$

For every $x_{ij} \in \mathcal{X}$, $\exists \mathcal{X}'$ and $\mathcal{X}''$ such that

$$\text{(ii)} \qquad \sum_{k=1}^{j-1} w_k + \sum_{x_{i'j'} \in \mathcal{X}'} \sum_{k=1}^{j'} w_k \leq C \qquad \mathcal{X}' \subseteq \mathcal{X}, |\mathcal{X}'| = p, x_{ij} \notin \mathcal{X}'$$

$$\text{(iii)} \qquad \sum_{k=1}^{N} w_k + \sum_{x_{i'j'} \in \mathcal{X}''} \sum_{k=1}^{j'} w_k \leq C \qquad \mathcal{X}'' \subseteq \mathcal{X}, |\mathcal{X}''| = p-1, x_{ij} \notin \mathcal{X}''$$

Condition (i) indicates that no group S_i of $\mathcal{P}$ is empty. Condition (ii) stipulates that for any group i, there is a feasible solution selecting item $(i, j-1)$, $j \in S_i$ without selecting item (i, j), and $rank(\mathcal{P})$ items of $\mathcal{P}$ in groups different from i. Condition (iii) specifies that for any group i, there is a feasible solution selecting the last item (i, M) of group i and $rank(\mathcal{P}) - 1$ items of $\mathcal{P}$ in groups different from i. It can be shown that (i), (ii) and (iii) can be checked in polynomial time. Moreover, we prove that verifying (ii) for 2 items and (iii) for 1 item is sufficient to verify (ii) and (iii). We will show how to obtain facet-defining conditions for more general patterns. Condition (i) is already a necessary facet-defining condition for any pattern. Conditions (ii) and (iii) can be generalized for any pattern, and can still be checked in polynomial time. It is proven that a pattern $\mathcal{P}$ with one set S_i of cardinality 1, and no restriction on other sets, leads to facet-defining inequalities of the SWMK if and only if (i) and generalized conditions corresponding to (ii) and (iii) hold.

References

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