A particular Quadratic Transportation Problem

Davide Duma[1], Stefano Gualandi[1], and Federico Malucelli[2]

[1] Università degli Studi di Pavia, Dipartimento di Matematica "F. Casorati"
via A. Ferrata 5, 27100 Pavia, Italy
{davide.duma, stefano.gualandi}@unipv.it
[2] Politecnico di Milano, Dipartimento di Elettronica e Informazione
Piazza L. da Vinci 32, 20133 Milano, Italy
federico.malucelli@polimi.it

Keywords: Transportation Polytope, Quadratic Programming, Reformulations

1 Introduction

The Quadratic Transportation Problem (QTP) is an optimization problem derived from the well known Transportation Problem (TP), where the objective function is quadratic in the flow variables. While the TP has a combinatorial nature and efficient combinatorial optimization algorithms have been developed, the quadratic objective function of QTP breaks this structure. Indeed, the main solution approaches for QTP are of continuous nature, mainly based on Lagrangean relaxations [1].

In this paper, we consider a particular QTP where the quadratic coefficients depend on the supply and request levels of the transportation problem [2]. This particular case of QTP arises in a statistical application that are described as follows. Given a set of marginal frequencies a_i, $i = 1, \dots, m$, and b_j, $j = 1, \dots, n$, we want to find the joint distribution x_{ij} that maximizes the χ^2 index, over a given Fréchet class $F(a, b)$ (for the details, see [2]). This is formulated as follows:

$$\chi^2 := \max \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{a_i b_j} x_{ij}^2 \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad i = 1, \dots, m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1, \dots, n \tag{3}$$

$$0 \le x_{ij} \le u_{ij} = \min\{a_i, b_j\} \qquad i = 1, \dots, m, j = 1, \dots, n. \tag{4}$$

Since a_i and b_j are positive numbers, the problem consists in maximizing a convex function over a convex set, which is, in principle, a difficult problem. Due to the convexity of the objective function, the optimal solution is an extreme point of the feasible region. In [2], three combinatorial heuristics are proposed to generate good extreme points of the transportation polytope, along with a Frank-Wolfe approach to compute an upper bound (UB).

In this paper, we introduce a new method to compute tight upper bounds, based on the solutions of quadratic combinatorial subproblems. The combinatorial subproblems can be solved by combinatorial algorithm that exploits the cost structure of the problem.

2 An upper bound based on problem decomposition

Let α be a scalar, with $\alpha \in [0,1]$. For each original variable x_{ij}, we introduce a twin variable y_{ij} that must be equal to x_{ij}. Hence, we can rewrite problem (1)–(4) as follows

$$P(\alpha) := \max \quad \alpha \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{a_i b_j} x_{ij}^2 + (1-\alpha) \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{a_i b_j} y_{ij}^2 \tag{5}$$

$$(2), (4)$$

$$\sum_{i=1}^{m} y_{ij} = b_j \qquad j = 1, \ldots, n \tag{6}$$

$$0 \le y_{ij} \le u_{ij}, \quad i = 1, \ldots, m, j = 1, \ldots, n \tag{7}$$

$$x_{ij} = y_{ij}, \quad i = 1, \ldots, m, j = 1, \ldots, n. \tag{8}$$

Note that the value of the optimal solution of $P(\alpha)$ is not affected by the value of the parameter α. Moreover, we can prove the following result.

Theorem 1. *$P(\alpha)$ is equivalent to P for any choice of α.*

Let us consider the relaxation $\bar{P}(\alpha)$ of problem $P(\alpha)$ where constraints (8) are eliminated. Hence, $\bar{P}(\alpha)$ can be decomposed into two separable subproblems having the same structure:

$$P_A(\alpha) := \max \left\{ \alpha \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{a_i b_j} x_{ij}^2, \qquad \text{such that} \quad (2), (4) \right\},$$

$$P_B(\alpha) := \max \left\{ (1-\alpha) \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{1}{a_i b_j} y_{ij}^2, \quad \text{such that} \quad (6), (7) \right\}.$$

Notice that for any value of $\alpha \in [0,1]$, the sum of the optimal objective functions value of $P_A(\alpha)$ and $P_B(\alpha)$ give an upper bound to the original problem P. Since α is a parameter that multiplies the whole objective function, the best upper bound is given by the minimum between $P_A(1)$ and $P_B(0)$.

Let us consider problem $P_A(1)$. All properties can be immediately extended to $P_A(\alpha)$ and to $P_B(\alpha)$. Note that $P_A(1)$ is separable into m subproblems, since the constraints and the objective function terms are independent. Let us identify by P_{A_i} the i-th subproblem:

$$P_{A_i} := \max \left\{ \frac{1}{a_i} \sum_{j=1}^{n} \frac{1}{b_j} x_{ij}^2, \text{ such that } \sum_{j=1}^{n} x_{ij} = a_i, 0 \le x_{ij} \le u_{ij}, j = 1, \ldots, n \right\}.$$

A direct consequence of the objective function convexity are the following results.

Theorem 2. *The optimal solution of* P_{A_i} *has at most one component* $0 < x_{ih} < b_h$ *and all the other components* x_{ij} *equal to 0 or* b_j.

Corollary 1. *If* $\tilde{j} = argmin\{b_j \mid b_j > a_i\}$ *exists, then the optimal solution of* P_{A_i} *has* $x_{ij} = 0$ *for all* j *s.t.* $b_j > b_{\tilde{j}}$.

Interestingly, the solution of problem P_{A_i} is indeed nontrivial.

Theorem 3. P_{A_i} *is NP-hard.*

More importantly, we can show that problem P_{A_i} can be solved by a simple combinatorial method. The pseudocode is reported in Algorithm 2, where, without loss of generality, we have supposed $b_1 \leq b_2 \leq \ldots \leq b_n$.

Algorithm 1 Combinatorial algorithm for solving P_{A_i}

$bestval \leftarrow 0$;
$n' = \text{argmin}\{b_j \mid b_j > a_i\}$
for $h = 1, \ldots, n'$ **do**
  $S \leftarrow \{1, \ldots, n'\} \setminus \{h\}$
  $val \leftarrow \text{BestSol}(a_i, b, S, h)$
  if $val > bestval$ **then**
    $bestval \leftarrow val$;
  end if
end for
return $bestval$;

For each h such that x_{ih} can not be fixed to 0 exploiting Corollary 1, the algorithm computes the best solution supposing that x_{ih} is the only component of Theorem 2 that can be neither 0 or b_h. Such a solution is provided by the function `BestSol`(a_i, b, S, h), which solves a combinatorial problem that can be reformulated as two 0–1 knapsack problems. The solution with the maximum value of the objective function among all $h \in S$ is the optimal solution of P_{A_i}.

A preliminary computational analysis has been performed on several tests using as benchmarks a set of random instances similar to those proposed in [2]. Results show the effectiveness of the proposed method for some patterns of the instances.

References

1. Adlakha, V., Kowalski, K.: On the Quadratic Transportation Problem. Open Journal of Optimization 2(3), 89–94 (2013).
2. Kalantari, B., Lari, I., Rizzi, A., Simeone, B.: Sharp bounds for the maximum of the chi-square index in a class of contingency tables with given marginals. Computational Statistics & Data Analysis 16, 19–34 (1993).