On computing the Edgeworth-Pareto hull for certain classes of multi-objective optimization problems*

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Extended abstract

We are interested in general multi-objective optimization problems of the following form and require further assumptions below:

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t. } x \in \mathcal{X}, \end{aligned}$$

where $f\colon \mathcal{X} \to \mathbb{R}^p$, $p \in \mathbb{N}$, and $\mathcal{X}$ is a set. For this problem we define $\mathcal{Q}^+ := \operatorname{cl}\operatorname{conv} f(\mathcal{X}) + \mathbb{R}^p_{\geq 0}$,the *Edgeworth-Pareto hull*. We observe that $\mathcal{Q}^+$ is a closed convex polyhedron. In our problem setting, we are interested in the facets of $\mathcal{Q}^+$. We make the following assumptions about the problem:

1. The value $\inf \left\{ w^\mathsf{T} f(x) : x \in \mathcal{X} \right\}$ is computable for every $w \in \mathbb{Q}^p_{\geq 0}$ and has finite value.
2. $\mathcal{Q}^+$ is finitely generated.

From its definition and (1) it follows that $\mathcal{Q}^+$ is a rational polyhedron, $\operatorname{rec} \mathcal{Q}^+ = \mathbb{R}^p_{\geq 0}$, and there is $y \in \mathbb{R}^p$ with $\mathcal{Q}^+ \subseteq y + \mathbb{R}^p_{\geq 0}$, i.e., an ideal point exists. The assumptions include *multi-objective mixed integer linear programming (MOMILP)*, and also certain classes of non-linear problems, e.g., multi-objective combinatorial problems with quadratic constraints and objective functions.

In this work, we present, to the best of our knowledge, the first *outer approximation (OA)* algorithm which can compute the Edgeworth-Pareto hull for MOMILP and the other classes of problems satisfying our assumptions.

* This research was funded in whole, or in part, by the Austrian Science Fund (FWF) [P 31366-NBL] and [P 35160-N]. For the purpose of open access, the author has applied a CC BY public copyright licence to any Author Accepted Manuscript version arising from this submission. It is also supported by the Johannes Kepler University Linz, Linz Institute of Technology (Project LIT-2019-7-YOU-211) and the JKU Business School.

Our research is motivated by recent advances in general purpose exact methods for bi- and multi-objective (mixed) integer linear programming relying on the branch-and-bound (B&B) idea. All recent successful implementations rely on lower bound sets [3], see, e.g., [6,8,4,1] but they are currently either restricted to two objectives or they only address the pure integer and not the mixed integer case. Lower bound sets are sets in the objective space. They are a natural multi-objective extension of the lower bounds obtained by, e.g., relaxations in single-objective optimization. In a bi-objective context, they are usually computed "from the inside" in so-called *inner approximation* (IA) schemes, see, e.g., [6,9]. Only [4,5] rely on Benson's algorithm, as implemented in `bensolve` [7] to compute bound sets "from the outside", using an OA scheme, in the context of a B&B for MOMILP. Benson's algorithm produces the Edgeworth-Pareto hull of the linear programming relaxation of the considered MOMILP. We observe that in the context of multi-objective B&B, our algorithm can directly compute the Edgeworth-Pareto hull of the MOMILP without the need of any relaxation.

Our algorithm relies on an *oracle* which solves single-objective *weighted-sum problems* and we show that the required number of oracle calls is polynomial in the number of facets of the convex hull of the non-dominated extreme points. As a consequence, for problems for which the weighted-sum problem is solvable in polynomial time, the facets can be computed in incremental polynomial time. These results extend and complement the results of [2] who showed that with an IA algorithm, the extreme points can be found in incremental polynomial time in this case. From a practical perspective, due to its OA nature, the algorithm starts from a valid lower bound set and iteratively improves it. Therefore it can be used in multi-objective B&B algorithms and still provide a valid bound set at any stage, even if interrupted before converging. We observe that for IA schemes this is not possible, i.e., IA schemes only produce valid lower bound sets when they are run until convergence. Moreover, the oracle produces Pareto optimal solutions, which makes the algorithm also attractive from the primal side. Finally, the oracle can also be called with any relaxation of the primal problem, and the obtained points and facets still provide a valid lower bound set. A computational study on a large set of benchmark instances, including non-linear instances, is conducted to assess the efficiency of our algorithm. We give empirical results on the numerical accuracy of our approach and provide a comparison with `PolySCIP`, which is an IA solver for MOMILP.

References

1. Adelgren, N., Gupte, A.: Branch-and-bound for biobjective mixed-integer linear programming. INFORMS Journal on Computing (2021)
2. Bökler, F., Mutzel, P.: Output-sensitive algorithms for enumerating the extreme nondominated points of multiobjective combinatorial optimization problems. In: Algorithms-ESA 2015, pp. 288–299. Springer (2015)
3. Ehrgott, M., Gandibleux, X.: Bound sets for biobjective combinatorial optimization problems. Computers & Operations Research **34**(9), 2674–2694 (2007)

4. Forget, N., Gadegaard, S.L., Nielsen, L.R.: Warm-starting lower bound set computations for branch-and-bound algorithms for multi objective integer linear programs. European Journal of Operational Research (2022)
5. Forget, N., Gadegaard, S.L., Nielsen, L.R., Klamroth, K., Przybylski, A.: Objective branching in branch and bound algorithms : the three objective case. Tech. rep., Optimization Online (2020)
6. Gadegaard, S.L., Nielsen, L.R., Ehrgott, M.: Bi-objective branch-and-cut algorithms based on lp relaxation and bound sets. INFORMS Journal on Computing **31**(4), 790–804 (2019)
7. Löhne, A., Weißing, B.: The vector linear program solver bensolve–notes on theoretical background. European Journal of Operational Research **260**(3), 807–813 (2017)
8. Parragh, S.N., Tricoire, F.: Branch-and-bound for bi-objective integer programming. INFORMS Journal on Computing **31**(4), 805–822 (2019)
9. Parragh, S.N., Tricoire, F., Gutjahr, W.J.: A branch-and-benders-cut algorithm for a bi-objective stochastic facility location problem. OR Spectrum pp. 1–41 (2021)