A fractional programming method for optimal assortment under the nested-logit model

Laurent Alfandari, Alborz Hassanzadeh, and Ivana Ljubic[1]

ESSEC Business School, 95021 Cergy-Pontoise, France,
name@essec.edu

Abstract. We study the problem of finding an assortment of products maximizing the expected revenue, in which customer preferences are modeled by a nested logit choice model. This problem is polynomially solvable in a specific case and NP-hard otherwise. We provide an exact general method that embeds a tailored Branch-and-Bound algorithm into a fractional programming framework. In contrast to the existing literature, no assumptions on the input data are imposed. Although our approach is not polynomial in input size, it can solve the most general problem setting for large-size instances. We show that the fractional programming parameterized subproblem, a highly non-linear binary optimization problem, is decomposable by nests. To solve the subproblem for each nest, we propose a two-stage approach. In the first stage, we fix a large set of variables based on the single-nest subproblems newly-derived structural properties. In the second stage, we design a tailored Branch-and-Bound algorithm with specific upper bounds. Our approach can solve instances with five nests and up to 5000 products per nest.

Keywords: assortment, choice model, combinatorial optimization, fractional programming, non-linear programming.

1 Introduction

Assortment optimization is the problem of choosing a portfolio of products to offer to customers to maximize the expected revenue. This class of problems has important applications in retail, online advertising, revenue management, etc. ([3]). An extension of random utility models and Multinomial Logit (MNL) choice models is the Nested Logit (NL) model. The general idea of this model is that customers first select a nest (i.e., a category of products), and then choose a product within that nest. As a concrete example, consider a tourist who is using online search engines to book a hotel room. In this case, the nests are the available hotels, and a product is a room within each hotel. The benchmark paper for our research about the Assortment Optimization Problem under the Nested-Logit model (AOPNL) is the paper of [1], which proposes approximation algorithms for the NP-hard cases.

2 Problem statement

We note $M = \{1, \dots, m\}$ the set of all nests, and $N = \{1, \dots, n\}$ is the set of products. We call $\mathcal{C}_i$ the collection of all feasible assortments for nest $i \in M$, and an assortment in nest i is denoted by $S_i \in \mathcal{C}_i$. $r_{ik} \geq 0$ is the revenue obtained by selling one unit of product $k \in S_i$. Customers' preference for product k in nest i is v_{ik}. Assuming a customer chooses nest i and assortment $S_i \in \mathcal{C}_i$ is offered for nest i, the probability that a customer chooses product k is:

$$P_{ik}(S_i) = \frac{v_{ik}}{v_{i0} + \sum_{k \in S_i} v_{ik}}$$

if $k \in S_i$, 0 otherwise, where v_{i0} is the attractiveness of leaving nest i without any purchase. Furthermore, the probability that a particular nest i is chosen given assortment $(S_1, \dots, S_m) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_m$ is offered, is:

$$Q_i(S_1, \dots, S_m) = \frac{(v_{i0} + \sum_{k \in S_i} v_{ik})^{\gamma_i}}{V_0 + \sum_{l \in M} (v_{l0} + \sum_{k \in S_l} v_{lk})^{\gamma_l}} \tag{1}$$

where V_0 is the attractiveness of the no-purchase option and γ_i is the dissimilarity parameter of nest i. Finally, the assortment optimization problem where customer choice behavior follows a nested logit model is:

$$\begin{aligned}
\max_{(S_1,\dots,S_m), S_i \in \mathcal{C}_i, i \in M} Z &= \Pi(S_1, \dots, S_m) \\
&= \sum_{i \in M} Q_i(S_1, \dots, S_m) \sum_{k=1}^{n} P_{ik}(S_i) r_{ik} \\
&= \sum_{i \in M} \frac{(v_{i0} + \sum\limits_{k \in S_i} v_{ik})^{\gamma_i}}{V_0 + \sum\limits_{l \in M} (v_{l0} + \sum\limits_{k \in S_l} v_{lk})^{\gamma_l}} \times \frac{\sum\limits_{k \in S_i} r_{ik} v_{ik}}{v_{i0} + \sum\limits_{k \in S_i} v_{ik}} \\
&= \frac{\sum\limits_{i \in M} (v_{i0} + \sum\limits_{k \in S_i} v_{ik})^{\gamma_i - 1} \sum\limits_{k \in S_i} r_{ik} v_{ik}}{V_0 + \sum\limits_{i \in M} (v_{i0} + \sum\limits_{k \in S_i} v_{ik})^{\gamma_i}} \qquad (2)
\end{aligned}$$

[1] proved that unless $\gamma_i \leq 1$ and $v_{i0} = 0$ for all $i \in M$, the problem AOPNL is NP-hard. We can hoewever use fractional programming to find the optimal solution of this problem even when $\gamma_i > 1$ or $v_{i0} > 0$.

3 A fractional programming approach

Assume we can enumerate all 2^n subsets (possible assortments) in each collection $\mathcal{C}_i$, for $i \in M$. Define a binary variable $x_{S_i} = 1$, iff we offer assortment $S_i \in \mathcal{C}_i$. Then problem AOPNL can be rewritten as:

$$\max \frac{\sum_{i\in M}\sum_{S_i\in\mathcal{C}_i}\left((v_{i0}+\sum_{k\in S_i}v_{ik})^{\gamma_i-1}\sum_{k\in S_i}r_{ik}v_{ik}\right)x_{S_i}}{V_0+\sum_{i\in M}\sum_{S_i\in\mathcal{C}_i}\left((v_{i0}+\sum_{k\in S_i}v_{ik})^{\gamma_i}\right)x_{S_i}} = \frac{N(x)}{D(x)} \tag{3}$$

$$\text{s.t:} \quad \sum_{S_i\in\mathcal{C}_i} x_{S_i} = 1 \qquad \forall i \in M$$

$$x_{S_i} \in \{0,1\}, \qquad \forall S_i \in \mathcal{C}_i, i \in M$$

Which is a linear fractional (or hyperbolic) program in variables x_{S_i}. Following [2], [4] proposed a method to solve fractional programs with 0-1 variables. The idea is to iteratively solve a so-called *parameterized* problem $F_{par}(\lambda) = N(x) - \lambda D(x)$ until $|N(x) - \lambda D(x)| = 0$. Then the λ^* found is the optimal ratio, and x^* is the vector optimizing the ratio. For the assortment optimization problem, the corresponding iterative *parameterized problem* $(F_{par}(\lambda))$ is:

$$\max \sum_{i\in M}\sum_{S_i\in\mathcal{C}_i}\left((v_{i0}+\sum_{k\in S_i}v_{ik})^{\gamma_i-1}\sum_{k\in S_i}r_{ik}v_{ik}\right)-\lambda\left(V_0+\sum_{i\in M}\sum_{S_i\in\mathcal{C}_i}(v_{i0}+\sum_{k\in S_i}v_{ik})^{\gamma_i}\right)$$

We use a parametric search to find the optimal assortment for the original problem, exploiting the fact that $F_{par}(\lambda)$ is piece-wise linear in λ. We can easily see that the above parametrized problem can be decomposed by nest i. For solving each subproblem at each iteration of the parametrized scheme, we design a tailored branch-and-bound method based on problem-specific upper bounds of the objective function (remind that the subproblem remains highly non-linear). Reduction techniques are first used to fix some products in the optimal assortment, based on revenues and utilities. Details will be provided at the conference. We test our method on instances with five nests and up to 5000 products, that we can solve to optimality whereas standard non-linear solvers fail even for small instances. The most challenging instances are the ones with a mix of dissimilarity parameters $\gamma_i < 1$ and $\gamma_i \geq 1$ over the nests.

References

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