# Generalized Relax-and-Fix heuristic

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**Abstract.** This paper introduces a heuristic for mixed-integer mathematical programs, that can be seen as a generalization of the relax-and-fix heuristic: a sequence of derived subproblems is solved, progressively fixing variables in the original problem. We propose a generic implementation and report on numerical results for four well-known operational research applications: lot-sizing, vehicle routing, bin-packing and portfolio optimization. Results show that this heuristic may be competitive depending on the definition of subproblems.

**Keywords:** Mixed-integer linear and non linear programming, greedy heuristic, relaxed problem

## 1 Literature

Relax-and-fix is a general heuristic to solve mixed-integer programs. It is described in [4] and consists in the following procedure. First, the set of integer variables is partitioned into disjunctive subsets, and a processing order is defined. Then, at each iteration, integrity constraints are relaxed for all but one of these subsets. The resulting subproblem is solved. After solving, the integer variables of the subproblem are fixed at their current values and the process is repeated for all the remaining subsets. Relax-and-fix has been successfully applied to multi-periodic scheduling and production planning problems. The authors of [3] propose an improvement of the Relax-and-Fix solution by a Fix-and-Optimize procedure: at each iteration, all variables except a small group are fixed to the value of the current solution and the free variables are optimized. The method improving the Relax-and-Fix solution is called exchange in [4]. In the context of the vehicle routing problem, studies can be found in the naval field [2, 1].In these studies, Relax-and-Fix tends to require too much time to solve large instances. Variants are then proposed. The basic idea of these variants is the omission of variables and their associated constraints. In [1], the Fix-and-Optimize algorithm aims to improve the solution obtained by the Relax-and-Fix. We propose a generalization of this heuristic which takes advantage of the inherent decomposition of the problems: in addition to relaxing some integrality constraints, the problem is restricted by relaxing other constraints. We formalize this heuristic, develop a generic software library, and evaluate its adaptability to diverse problems.

## 2 Restrict-and-Fix Heuristic

For the next, consider the problem $\max\{c{\cdot}x : Ax \leq b, x \in D\}$, where $c$ is a vector of size $n$, $A$ is a matrix of size $m \times n$, $b$ is a vector of size $m$ and $D = \mathbb{Z}^p \times \mathbb{R}^q$ is the variables space with $p + q = n$. Moreover, for each $I \subset \{1, \ldots, m\}$ and $J \subset \{1, \ldots, n\}$, let $A[I, J]$ the submatrix of $A$ composed of all coefficients of $A$ with their row indexes in $I$ and column indexes in $J$, $c[J]$ is the subvector of $c$ composed of all components of $c$ with their indexes in $J$, $b[I]$ is the subvector of $b$ composed of all components of $b$ with their indexes in $I$ and $D[J]$ is the subspace of $D$ corresponding to variable indexes in $J$.

The Restrict-and-Fix algorithm consists in solving a sequence of smaller subproblems, derived from the formulation, in order to progressively construct a solution to the original problem, by fixing variables to their values in the subproblems' solutions. This sequence is based on a partition of variables into $K$ subsets of *local* variables, each subset defining one subproblem and another set of *global* variables present in all subproblems. Let $(J_k)_{k=0,\ldots,K}$ a $K$-partition of $\{1, \ldots, n\}$, the set of variable indexes. The set $J_0$ corresponds to global variable indexes and the sets $J_1, \ldots, J_K$ correspond to *blocks* of variable indexes associated to one specific *smaller* subproblem. For each $k \in \{1, \ldots, K\}$, each constraint which contains only local variables associated to the block $J_k$ is called a *local* constraint. Other constraints are called *global* constraints. Let $(I_k)_{k=0,\ldots,K}$ the partition of $\{1, \ldots, m\}$, where $I_0$ is the set of global constraint indexes and, for each $k \in \{1, \ldots, K\}$, $I_k$ is the set of local constraint indexes associated to the block $J_k$. A *smaller* subproblem is defined for each *block* $J_k$ composed of all local variables of $J_k$, some global variables, local constraints associated to $J_k$ and all global constraints containing at least one local or one global variable.

So, at each iteration $k \in \{1, \ldots, K\}$ of the Restrict-and-Fix algorithm, the following *smaller subproblem* $[SP_k]$ *associated to the block* $J_k$ is defined and solved to fix some local variables:

$$
\begin{aligned}
[SP_k]\ \max\quad & c[J_k] \cdot x[I_k] + \quad c[I_0] \cdot x[J_0] && && (1)\\
\text{s.t. } & A[I_k, J_k]\, x[J_k] && \leq b[I_k] && (2)\\
& A[I_0, J_k]\, x[J_k] + A[I_0, J_0]\, x[J_0] && \leq b[I_0] - \sum_{k'=1}^{k-1} A[I_0, J_{k'}]\, \hat{x}[J_{k'}] && (3)\\
& x[J_k] \in D[J_k] \ \ , \ \ x[J_0] \in D[J_0] && && (4)
\end{aligned}
$$

where $\hat{x}$ correspond to values of the previously fixed variables.

To ensure existence of a feasible solution, partial global constraints may be by-passed (some global constraints (3) can be deleted in the subproblem $k$ because they can be infeasible if local variables for blocks $k' > k$ are not considered) or may be completed with artificial variables. Moreover, for more flexibility, the bounds of global variables, their types and their coefficients in objective may be changed. he bound modifications on $x[J_0]$ allows to disable some global variables and thus disable some global constraints (3). After solving subproblem $[SP_k]$, variables $x[I_k]$ are fixed for the next subproblems to their values in the best found solution.

At the end of the Restrict-and-Fix algorithm, the following *residual subproblem* is defined and solved to fix remaining variables:

$$[SP_{res}] \quad \max \quad c[I_0] \cdot x[J_0] \tag{5}$$

$$\text{s.t.} \quad A[I_0, J_0]\, x[J_0] \ \leq \ b[I_0] - \sum_{k'=1}^{K} A[I_0, J_{k'}]\, \hat{x}[J_{k'}] \tag{6}$$

$$x[J_0] \in D[J_0] \tag{7}$$

After solving the residual subproblem, variables $x[I_0]$ are fixed thanks to the best found solution.

## 3 Numerical experiments

The proposed implementation is generic and adaptable for various classes of problems thanks to the options. As for other greedy heuristics, the quality or the feasibility cannot be guaranteed. In this case, a generic local search algorithm based on the concept of Fix-and-Optimize has been proposed in order to improve the obtained solutions or to try to reach feasibility. The basic idea is to unfix some variables and to solve the residual problem. Restrict-and-Fix is compared to the no-partition approach and to the Relax-and-Fix. Numerical results on four well-known applications (CMILSP, VRP, BPP, Portofolio problem) show the competitiveness of Restrict-and-Fix, especially on large instances. In addition, it appeared that the results of Restrict-and-Fix are more stable when the size of instances increases. Moreover, the proposed implementations for VRP and BPP are easily adapted to their variants VRPTW and CSP and results remain competitive.

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